

Boundedness and stabilization in a two-species chemotaxis-competition system of parabolic-parabolic-elliptic type

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Abstract. This paper deals with the two-species chemotaxis-competition system

$$\begin{cases} u_t = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v) & \text{in } \Omega \times (0, \infty), \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v) & \text{in } \Omega \times (0, \infty), \\ 0 = d_3 \Delta w + \alpha u + \beta v - \gamma w & \text{in } \Omega \times (0, \infty), \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$, $n \geq 2$; χ_i and μ_i are constants satisfying some conditions. The above system was studied in the cases that $a_1, a_2 \in (0, 1)$ and $a_1 > 1 > a_2$, and it was proved that global existence and asymptotic stability hold when $\frac{\chi_i}{\mu_i}$ are small ([5, 32, 34]). However, the conditions in the above two cases strongly depend on a_1, a_2 , and have not been obtained in the case that $a_1, a_2 \geq 1$. Moreover, convergence rates in the cases that $a_1, a_2 \in (0, 1)$ and $a_1 > 1 > a_2$ have not been studied. The purpose of this work is to construct conditions which derive global existence of classical bounded solutions for all $a_1, a_2 > 0$ which covers the case that $a_1, a_2 \geq 1$, and lead to convergence rates for solutions of the above system in the cases that $a_1, a_2 \in (0, 1)$ and $a_1 \geq 1 > a_2$.

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1. Introduction

Many phenomena, which appear in natural science, especially, biology, chemistry and physics, support animals' lives. In this paper we focus on *chemotaxis* which is one of the important properties and is related to e.g., movement of sperm, migration of neurons or lymphocytes and tumor invasion. Chemotaxis is the property such that species move towards higher concentration of the chemical substance when they plunge into hunger.

A mathematical problem which describes a part of the life cycle of cellular slime molds with chemotaxis is called the Keller–Segel system:

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \quad \tau v_t = \Delta v + u - v,$$

where $\chi > 0$ and $\tau \in \{0, 1\}$. Moreover, the chemotaxis system with growth terms

$$u_t = \Delta u + \chi \nabla \cdot (u \nabla v) + \kappa u - \mu u^2, \quad \tau v_t = \Delta v + u - v$$

was proposed by [25, 31], where $\chi, \kappa, \mu > 0$ and $\tau \in \{0, 1\}$. After the pioneering work of Keller–Segel [19], the Keller–Segel system and the chemotaxis system are intensively studied (see e.g., [2, 13, 15]). A generalized problem of Keller–Segel systems, which means a two-species chemotaxis system, was proposed in [36] and also has studied (see e.g., [3, 4, 7, 8, 10, 21, 37]; global existence was proved in [7, 8, 37]; and thier asymptotic stability was shown in [37]; related works which deal with blow-up of solutions can be seen in [3, 4, 7, 8, 10, 21]). Recently, a two-species chemotaxis system with competitive kinetics

$$\begin{aligned} u_t &= \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), \\ v_t &= \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), \\ \tau w_t &= \Delta w + \alpha u + \beta v - \gamma w \end{aligned}$$

with some $\chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2 > 0$ and $\tau \in \{0, 1\}$, which describes the evolution of two competing species which react on a single chemoattractant, was proposed by Tello–Winkler [34] and was studied (see [1, 22, 26, 27, 28, 29, 30, 38]). About this problem with $\tau = 1$, global existence and boundedness was obtained in the 2-dimensional case ([1]) and the n -dimensional setting ([22]); moreover, asymptotic behavior of solutions was established in [1, 27]. Related works which dealt with global existence and boundedness in this two-species problem with sensitivity functions can be found in [27, 38]; and related works which treated the non-competition case are in [26, 28, 29, 30]. These results in the case $\tau = 1$ are motivated by the results ([5, 32, 34]) in the case $\tau = 0$. Therefore the parabolic-parabolic-elliptic problem reduced by letting $\tau = 0$ seems to be helpful to analyze the fully parabolic case.

In this paper we consider the two-species chemotaxis system with competitive kinetics of parabolic-parabolic-elliptic type

$$\begin{cases} u_t = d_1 \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega, \ t > 0, \\ v_t = d_2 \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - a_2 u - v), & x \in \Omega, \ t > 0, \\ 0 = d_3 \Delta w + \alpha u + \beta v - \gamma w, & x \in \Omega, \ t > 0, \\ \nabla u \cdot \nu = \nabla v \cdot \nu = \nabla w \cdot \nu = 0, & x \in \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x), \ v(x, 0) = v_0(x), & x \in \Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^n ($n \geq 2$) with smooth boundary $\partial\Omega$ and ν is the outward normal vector to $\partial\Omega$. The constants $d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2$ and α, β, γ are positive. The initial data u_0, v_0 are assumed to be nonnegative functions. The unknown functions $u(x, t)$ and $v(x, t)$ represent the population densities of two species and $w(x, t)$ shows the concentration of the chemical substance at place x and time t .

The problem (1.1) is a problem on account of the influence of chemotaxis, diffusion, and the Lotka–Volterra competitive kinetics, i.e., with coupling coefficients $a_1, a_2 > 0$ in

$$u_t = u(1 - u - a_1 v), \quad v_t = v(1 - a_2 u - v). \quad (1.2)$$

The mathematical difficulties of the problem (1.1) are to deal with the chemotaxis term $\nabla \cdot (u \nabla w)$ and the competition term $u(1 - u - a_1 v)$. To overcome these difficulties, in the case that $a_1, a_2 \in (0, 1)$ and $d_3 = \alpha = \beta = 1$ in (1.1), Tello–Winkler [34] applied comparison methods to this problem and obtained global existence of classical bounded solutions and their asymptotic behavior under the conditions that

$$2(\chi_1 + \chi_2) + a_2 \mu_1 < \mu_2 \quad \text{and} \quad 2(\chi_1 + \chi_2) + a_1 \mu_2 < \mu_1. \quad (1.3)$$

However, if $\chi_1 \rightarrow 0$ or $\mu_1 \rightarrow \infty$, then these conditions break down. Recently, it was shown that the conditions

$$\frac{\chi_1}{\mu_1} \in \left[0, \frac{d_3}{2\alpha}\right) \cap \left[0, \frac{a_1 d_3}{\beta}\right), \quad \frac{\chi_2}{\mu_2} \in \left[0, \frac{d_3}{2\beta}\right) \cap \left[0, \frac{a_2 d_3}{\alpha}\right), \quad (1.4)$$

$$a_1 a_2 d_3^2 < \left(d_3 - \frac{2\alpha\chi_1}{\mu_1}\right) \left(d_3 - \frac{2\beta\chi_2}{\mu_2}\right) \quad (1.5)$$

lead to global existence and asymptotic stability in (1.1) in the case that $a_1, a_2 \in (0, 1)$ ([5]). The conditions (1.4)–(1.5) partially relax (1.3) in view of the point mentioned above. On the other hand, in the case that $a_1 > 1 > a_2$ and $d_3 = \beta = 1$ in (1.1) Stinner–Tello–Winkler [32] established global existence and stabilization of global classical solutions when

$$\begin{aligned} \frac{\chi_1}{\mu_1} &\leq a_1, \quad \frac{\chi_2}{\mu_2} < \frac{1}{2} \quad \text{and} \\ \frac{\alpha\chi_1}{\mu_1} + \max \left\{ \frac{\chi_2}{\mu_2}, \frac{a_2(\mu_2 - \chi_2)}{\mu_2 - 2\chi_2}, \frac{(\alpha - a_2)\chi_2}{\mu_2 - 2\chi_2} \right\} &< 1 \end{aligned}$$

are satisfied. In summary the two-species chemotaxis-competition model (1.1) were studied in the cases that $a_1, a_2 \in (0, 1)$ and $a_1 > 1 > a_2$, and it was proved that global existence and same asymptotic behavior as solutions to the Lotka–Volterra competition model (1.2) hold when $\frac{\chi_i}{\mu_i}$ are small. However, the conditions in the above two cases strongly depend on a_1, a_2 , and have not been obtained in the case that $a_1, a_2 \geq 1$. Moreover, convergence rates in the cases that $a_1, a_2 \in (0, 1)$ and $a_1 > 1 > a_2$ have not been studied.

The purpose of this work is to construct conditions which derive global existence of classical bounded solutions for all $a_1, a_2 > 0$ which covers the case that $a_1, a_2 \geq 1$, and lead to convergence rates for solutions of (1.1) in the cases that $a_1, a_2 \in (0, 1)$ and $a_1 > 1 > a_2$.

For establishing global existence and boundedness we shall suppose that χ_1, χ_2 and μ_1, μ_2 satisfy the following conditions:

$$\frac{\chi_1}{\mu_1} < \frac{nd_3}{n-2} \min \left\{ \frac{1}{\alpha}, \frac{a_1}{\beta} \right\} \quad \text{and} \quad \frac{\chi_2}{\mu_2} < \frac{nd_3}{n-2} \min \left\{ \frac{1}{\beta}, \frac{a_2}{\alpha} \right\}. \quad (1.6)$$

We assume that the initial data u_0, v_0 satisfy

$$0 \leq u_0 \in C(\overline{\Omega}) \setminus \{0\}, \quad 0 \leq v_0 \in C(\overline{\Omega}) \setminus \{0\}. \quad (1.7)$$

Now the main results read as follows. The first one is concerned with global existence and boundedness in (1.1).

Theorem 1.1. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 > 0$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that (1.6) are satisfied. Then for any u_0, v_0 satisfying (1.7) with some $q > n$, there exists an exactly one pair (u, v, w) of nonnegative functions*

$$\begin{aligned} u, v &\in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)), \\ w &\in C(\overline{\Omega} \times [0, \infty)) \cap C^{2,1}(\overline{\Omega} \times (0, \infty)) \cap L_{\text{loc}}^\infty([0, \infty); W^{1,q}(\Omega)), \end{aligned}$$

which satisfy (1.1). Moreover, the solutions u, v, w are uniformly bounded, i.e., there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all } t \geq 0,$$

and the solutions u, v, w are the Hölder continuous functions, i.e., there exist $\theta \in (0, 1)$ and $M > 0$ such that

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} \leq M \quad \text{for all } t \geq 1.$$

Remark 1.1. This result give the existence of global classical bounded solutions in the case that $a_1, a_2 \geq 1$. Moreover, the condition (1.6) relaxes (1.4) which assumed for global existence of solutions in [5]. Indeed, if χ_1, χ_2 and μ_1, μ_2 satisfy the condition (1.4), then χ_1, χ_2 and μ_1, μ_2 satisfy the condition (1.6). However, the condition (1.6) does not always relax those assumed in [32] and [34]; in the case that $a_1, a_2 \in (0, 1)$ the condition (1.6) relaxes (1.3) under the condition

$$\chi_1 < \frac{2na_1(\chi_1 + \chi_2)(1 + a_1)}{(n-2)(1 - a_1a_2)} \quad \text{and} \quad \chi_2 < \frac{2na_2(\chi_1 + \chi_2)(1 + a_2)}{(n-2)(1 - a_1a_2)},$$

and in the case that $a_1 > 1 > a_2$ the condition (1.6) relaxes the condition

$$\frac{\alpha\chi_1}{\mu_1} + \frac{\chi_2}{\mu_2} < 1,$$

which was used to obtain global existence in [32], when

$$\frac{\alpha(n-2)}{n} < \min\{1, a_1\alpha, a_2\}$$

hold.

The main theorem tells us the following result in the 2-dimensional case.

Corollary 1.2. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 > 0$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then for any u_0, v_0 satisfying (1.7) with some $q > n$, (1.1) possesses a unique global bounded classical solution.*

In the case $a_1, a_2 \in (0, 1)$ asymptotic behavior of solutions to (1.1) will be discussed under the following additional conditions: there exists $\delta_1 > 0$ such that

$$4\delta_1 - a_1 a_2 (1 + \delta_1)^2 > 0 \quad (1.8)$$

and

$$\mu_1 > \frac{\chi_1^2 (1 + \delta_1) (1 - a_1) (\alpha^2 a_1 \delta_1 + \beta^2 a_2 - \alpha \beta a_1 a_2 (1 + \delta_1))}{4a_1 d_1 d_3 \gamma (1 - a_1 a_2) (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}, \quad (1.9)$$

$$\mu_2 > \frac{\chi_2^2 (1 + \delta_1) (1 - a_2) (\alpha^2 a_1 \delta_1 + \beta^2 a_2 - \alpha \beta a_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma (1 - a_1 a_2) (4\delta_1 - a_1 a_2 (1 + \delta_1)^2)}. \quad (1.10)$$

The second theorem gives asymptotic behavior in (1.1) in the case $a_1, a_2 \in (0, 1)$.

Theorem 1.3. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 \in (0, 1)$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that there exists a unique global classical solution (u, v, w) of (1.1) satisfying*

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \leq M \quad \text{for all } t > 0$$

with some $M > 0$. Then under the conditions (1.8)–(1.10), (u, v, w) satisfies that there exist $C > 0$ and $\ell > 0$ such that

$$\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \leq C e^{-\ell t} \quad \text{for all } t > 0,$$

where

$$u^* := \frac{1 - a_1}{1 - a_1 a_2}, \quad v^* := \frac{1 - a_2}{1 - a_1 a_2}, \quad w^* := \frac{\alpha u^* + \beta v^*}{\gamma}.$$

Remark 1.2. If the assumption of Theorem 1.1 and (1.8)–(1.10) are satisfied, then Theorem 1.3 gives the convergence rates for solutions of (1.1) in the case that $a_1, a_2 \in (0, 1)$. Moreover, the conditions (1.8)–(1.10) are the same conditions as that assumed in [27] in the case that $a_1, a_2 \in (0, 1)$ and $h(u, v, w) = \alpha u + \beta v - \gamma w$.

In the case $a_1 \geq 1 > a_2$ asymptotic behavior of solutions to (1.1) will be discussed under the following additional conditions: there exist $\delta_1 > 0$ and $a'_1 \in [1, a_1]$ such that

$$4\delta_1 - a'_1 a_2 (1 + \delta_1)^2 > 0, \quad (1.11)$$

$$\mu_2 > \frac{\chi_2^2 \delta_1 (\alpha^2 a'_1 \delta_1 + \beta^2 a_2 - \alpha \beta a'_1 a_2 (1 + \delta_1))}{4a_2 d_2 d_3 \gamma (4\delta_1 - a'_1 a_2 (1 + \delta_1)^2)}. \quad (1.12)$$

The third one gives asymptotic behavior in (1.1) in the case $a_1 \geq 1 > a_2$.

Theorem 1.4. Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1 \geq 1$, $a_2 \in (0, 1)$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with smooth boundary. Assume that there exists a unique global classical solution (u, v, w) of (1.1) such that

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} \leq M \quad \text{for all } t > 0$$

with some $M > 0$. Then under the conditions (1.11)–(1.12), (u, v, w) has the following properties:

- (i) If $a_1 > 1$ and take $a'_1 \in (1, a_1]$ in (1.11)–(1.12), then there exist $C > 0$ and $\ell > 0$ satisfying

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \left\| w(t) - \frac{\beta}{\gamma} \right\|_{L^\infty(\Omega)} \leq C e^{-\ell t} \quad \text{for all } t > 0.$$

- (ii) If $a_1 = 1$, then there exist $C > 0$ and $\ell > 0$ satisfying

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t) - 1\|_{L^\infty(\Omega)} + \left\| w(t) - \frac{\beta}{\gamma} \right\|_{L^\infty(\Omega)} \leq C(t+1)^{-\ell} \quad \text{for all } t > 0.$$

Remark 1.3. If the assumption of Theorem 1.1 and (1.11)–(1.12) are satisfied, then Theorem 1.4 gives the convergence rates for solutions in the cases that $a_1 > 1 > a_2$ and $a_1 = 1 > a_2$. Moreover, the conditions (1.11)–(1.12) are the same conditions as that assumed in [27] in the case that $a_1 \geq 1 > a_2$ and $h(u, v, w) = \alpha u + \beta v - \gamma w$.

Remark 1.4. Stabilization in the case that $a_1, a_2 \geq 1$ is a still open question. In the case that $a_1, a_2 > 1$ a Lotka–Volterra competition model with diffusion term was studied; however, its analysis is difficult and it is known that solutions have complicated structures (see cf. [9, 16, 17, 18, 20, 23, 24]).

The strategy of the proof of Theorem 1.1 is to extend a method in [33] to a two-species case. We first aim to establish the L^p -estimate for u with some $p > \frac{n}{2}$ from the following derivative of $\int_\Omega u^p$:

$$\frac{1}{p} \frac{d}{dt} \int_\Omega u^p \leq (p-1) \chi_1 \int_\Omega u^{p-1} \nabla u \cdot \nabla v + \mu_1 \int_\Omega u^p (1 - u - a_1 v). \quad (1.13)$$

Since the third equation in (1.1) derives that

$$(p-1) \chi_1 \int_\Omega u^{p-1} \nabla u \cdot \nabla v = \frac{(p-1) \chi_1}{d_3 p} \int_\Omega u^p (\alpha u + \beta v - \gamma w), \quad (1.14)$$

we shall show that a combination of (1.13) and (1.14), along with the condition (1.6) implies

$$\frac{d}{dt} \int_\Omega u^p \leq -c_1 \left(\int_\Omega u^p \right)^{\frac{p+1}{p}} + c_2 \int_\Omega u^p,$$

which leads to L^p -estimate for u . Then aided by standard semigroup estimates, we can obtain the L^∞ -estimate for u . On the other hand, one of the keys for the proof of Theorems 1.3 and 1.4 is to derive the following energy estimate:

$$\frac{d}{dt}E(t) \leq -\varepsilon \int_{\Omega} [(u(\cdot, t) - \bar{u})^2 + (v(\cdot, t) - \bar{v})^2 + (w(\cdot, t) - \bar{w})^2] \quad (1.15)$$

for all $t > 0$ with some positive function E and some constant $\varepsilon > 0$, where $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^3$ is a solution of (1.1). Thanks to (1.15), we can obtain that there exists $C > 0$ such that

$$\int_0^\infty \int_{\Omega} (u - \bar{u})^2 + \int_0^\infty \int_{\Omega} (v - \bar{v})^2 + \int_0^\infty \int_{\Omega} (w - \bar{w})^2 \leq C,$$

which together with the regularity of the solution leads to Theorems 1.3 and 1.4.

This paper is organized as follows. In Section 2 we prove global existence and boundedness (Theorem 1.1) through a series of lemmas. Section 3 is devoted to the proof of asymptotic stability (Theorems 1.3 and 1.4); we first provide some lemmas which will be used later, and we next divide the section into Sections 3.1 and 3.2 according to the proof of Theorem 1.3 and that of Theorem 1.4, respectively.

2. Global existence and boundedness

In this section we shall show global existence and boundedness in (1.1). First we will recall the known result about local existence of solutions to (1.1) ([5, Lemma 2.1], [32, Lemma 2.1]).

Lemma 2.1. *Let $d_1, d_2, d_3 > 0$, $\mu_1, \mu_2 > 0$, $a_1, a_2 > 0$, $\chi_1, \chi_2 > 0$, $\alpha, \beta, \gamma > 0$ and let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a bounded domain with smooth boundary. Then for any u_0, v_0 satisfying (1.7) for some $q > n$, there exist $T_{\max} \in (0, \infty]$ and an exactly one pair (u, v, w) of nonnegative functions*

$$u, v, w \in C(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$$

which satisfy (1.1). Moreover,

$$\text{either } T_{\max} = \infty \quad \text{or} \quad \lim_{t \rightarrow T_{\max}} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}) = \infty.$$

We next give the L^p -estimate for u with some $p > \frac{n}{2}$ which plays an important role in deriving L^∞ -estimate for u . The proof is based on the proof of [33, Lemma 2.2].

Lemma 2.2. *Assume that (1.6)–(1.7) are satisfied. Then for all $p \in I_1$, there exists $C(p) > 0$ such that*

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq C(p)$$

for all $t > 0$, where

$$I_1 := \left(\frac{n}{2}, \min \left\{ \frac{\alpha\chi_1}{(\alpha\chi_1 - d_3\mu_1)_+}, \frac{\beta\chi_1}{(\beta\chi_1 - a_1d_3\mu_1)_+} \right\} \right).$$

Proof. We fix $p \in I_1$. Here we note from the condition (1.6) that $I_1 \neq \emptyset$. Multiplying the first equation in (1.1) by u^{p-1} and integrating it over Ω , we obtain that

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + d_1(p-1) \int_{\Omega} u^{p-2} |\nabla u|^2 \\ &= (p-1) \chi_1 \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \mu_1 \int_{\Omega} u^p (1 - u - a_1 v). \end{aligned} \quad (2.1)$$

Then integration by parts and the third equation in (1.1) imply that

$$\begin{aligned} (p-1) \chi_1 \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v &= -\frac{(p-1) \chi_1}{p} \int_{\Omega} u^p \Delta v \\ &= \frac{(p-1) \chi_1}{d_3 p} \int_{\Omega} u^p (\alpha u + \beta v - \gamma w). \end{aligned} \quad (2.2)$$

Therefore a combination of (2.1) with (2.2) yields that

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq \mu_1 \int_{\Omega} u^p - \left(\mu_1 - \frac{\alpha(p-1) \chi_1}{d_3 p} \right) \int_{\Omega} u^{p+1} - \left(a_1 \mu_1 - \frac{\beta(p-1) \chi_1}{d_3 p} \right) \int_{\Omega} u^p v.$$

Recalling $p \in I_1 = \left(\frac{n}{2}, \min \left\{ \frac{\alpha \chi_1}{(\alpha \chi_1 - d_3 \mu_1)_+}, \frac{\beta \chi_1}{(\beta \chi_1 - a_1 d_3 \mu_1)_+} \right\} \right)$ that

$$\mu_1 - \frac{\alpha(p-1) \chi_1}{d_3 p} > 0 \quad \text{and} \quad a_1 \mu_1 - \frac{\beta(p-1) \chi_1}{d_3 p} > 0,$$

we establish from the Hölder inequality

$$\int_{\Omega} u^p \leq |\Omega|^{\frac{p}{p+1}} \left(\int_{\Omega} u^{p+1} \right)^{\frac{p}{p+1}}$$

that there exists $\varepsilon > 0$ satisfying

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq -\varepsilon \left(\int_{\Omega} u^p \right)^{\frac{p+1}{p}} + \mu_1 \int_{\Omega} u^p,$$

which implies that

$$\|u(\cdot, t)\|_{L^p(\Omega)} \leq \min \left\{ \|u_0\|_{L^p(\Omega)}, \frac{\mu_1}{\varepsilon} \right\} \quad \text{for all } t \in (0, T_{\max}).$$

Thus we can attain the conclusion of this lemma. \square

Similarly, we can confirm the following L^p -estimate for v with some $p > \frac{n}{2}$.

Lemma 2.3. *Assume that (1.6)–(1.7) are satisfied. Then for all $p \in I_2$, there exists $C(p) > 0$ such that*

$$\|v(\cdot, t)\|_{L^p(\Omega)} \leq C(p) \quad \text{for all } t > 0,$$

where $I_2 := \left(\frac{n}{2}, \min \left\{ \frac{\beta \chi_2}{(\beta \chi_2 - d_3 \mu_2)_+}, \frac{\alpha \chi_2}{(\alpha \chi_2 - a_2 d_3 \mu_2)_+} \right\} \right)$.

Proof. A similar argument as in the proof of Lemma 2.3 derives this lemma. \square

Now we could construct all estimates which will enable us to obtain the estimate for the solution; Lemmas 2.2 and 2.3 lead to the following lemma. The proof is based on a known argument involving semigroup estimates which derive the L^∞ -estimate for u from L^p -estimate with $p > \frac{n}{2}$ (see e.g., [2]).

Lemma 2.4. *Assume that (1.6)–(1.7) are satisfied. Then there exists $C > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \quad \text{for all } t > 0. \quad (2.3)$$

Moreover, there exist $M > 0$ and $\theta \in (0, 1)$ such that

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{\theta}{2}}(\bar{\Omega} \times [t, t+1])} \leq M \quad \text{for all } t \geq 1.$$

Proof. We fix $p \in I_1 \cap I_2 \cap (0, n)$, where I_1 and I_2 are the intervals defined in Lemmas 2.2 and 2.3. Then thanks to Lemmas 2.2 and 2.3, we can find $C_1 > 0$ such that

$$\|u(\cdot, t)\|_{L^p(\Omega)} + \|v(\cdot, t)\|_{L^p(\Omega)} \leq C_1 \quad \text{for all } t \in (0, T_{\max}). \quad (2.4)$$

We first verify the $L^{\frac{np}{n-p}}$ -estimate for ∇v . Here for all $q \in (1, \infty)$, the standard elliptic regularity argument (see e.g., [11, Theorem 19.1]) leads to the existence of a constant $C_E(q) > 0$ satisfying

$$\|w(\cdot, t)\|_{W^{2,q}(\Omega)} \leq C_E(q)(\|u(\cdot, t)\|_{L^q(\Omega)} + \|v(\cdot, t)\|_{L^q(\Omega)}) \quad \text{for all } t \in (0, T_{\max}). \quad (2.5)$$

Therefore a combination of (2.5) with (2.4) yields from the Sobolev embedding theorem that there exists $C_2 > 0$ such that

$$\|\nabla v(\cdot, t)\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C_2 \quad \text{for all } t \in (0, T_{\max})$$

since $p < n$. We next establish the L^∞ -estimate for u . Since $p > \frac{n}{2}$, we can take $r \in (n, q)$ such that

$$p > \frac{nr}{n+r}.$$

We take $\vartheta > 1$ satisfying

$$\frac{1}{\vartheta} < \min \left\{ 1 - \frac{r(n-p)}{np}, \frac{q-r}{q} \right\}.$$

Then $\vartheta' := \frac{\vartheta}{\vartheta-1}$ satisfies

$$r\vartheta' < \frac{np}{n-p}.$$

Now for all $T \in (0, T_{\max})$ we note that

$$A(T) := \sup_{t \in (0, T)} \|u(\cdot, t)\|_{L^\infty(\Omega)}$$

is finite. To obtain the estimate for $A(T)$ we put $t_0 := (t-1)_+$ and represent u according to

$$\begin{aligned} u(\cdot, t) &= e^{(t-t_0)d_1\Delta}u(t_0) - \chi_1 \int_{t_0}^t e^{(t-s)d_1\Delta} \nabla \cdot (u(\cdot, s) \nabla v(\cdot, s)) ds \\ &\quad + \mu_1 \int_{t_0}^t e^{(t-s)d_1\Delta} u(\cdot, s) (1 - u(\cdot, s) - a_1 v(\cdot, s)) ds \\ &=: u_1(\cdot, t) + u_2(\cdot, t) + u_3(\cdot, t) \end{aligned} \quad (2.6)$$

for $t \in (0, T_{\max})$. In the case that $t \leq 1$, i.e., $t_0 = 0$, from the order preserving property of the Neumann heat semigroup we see that

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \quad \text{for all } t \in (0, 1] \cap (0, T). \quad (2.7)$$

In the case that $t > 1$ using the L^p - L^q estimate for $(e^{\tau\Delta})_{\tau \geq 0}$ (see [35, Lemma 1.3]) yields that there is $C_3 > 0$ such that

$$\|u_1(\cdot, t)\|_{L^\infty(\Omega)} \leq C_3 \|u(\cdot, t_0)\|_{L^p(\Omega)} \leq C_1 C_3 \quad \text{for all } t \in (1, T). \quad (2.8)$$

Next due to a known smoothing property of $(e^{\tau\Delta})_{\tau \geq 0}$ (see [12, Lemma 3.3]), we can find $C_4 > 0$ such that

$$\|u_2(\cdot, t)\|_{L^\infty(\Omega)} \leq C_4 \sup_{t \in (0, T)} \|u(\cdot, t) \nabla v(\cdot, t)\|_{L^r(\Omega)} \int_0^1 \sigma^{-\frac{1}{2} - \frac{n}{2r}} d\sigma.$$

Noting from $r\vartheta' < \frac{np}{n-p}$ and (2.4) that

$$\begin{aligned} \|u(\cdot, t) \nabla v(\cdot, t)\|_{L^r(\Omega)} &\leq \|u(\cdot, t)\|_{L^{r\vartheta}(\Omega)} \|\nabla v(\cdot, t)\|_{L^{r\vartheta'}(\Omega)} \\ &\leq C_5 \|u(\cdot, t)\|_{L^\infty(\Omega)}^{1-\frac{p}{r\vartheta}} \|u(\cdot, t)\|_{L^p(\Omega)}^{\frac{p}{r\vartheta}} \|\nabla v(\cdot, t)\|_{L^{\frac{np}{n-p}}(\Omega)} \\ &\leq C_1^{\frac{p}{r\vartheta}} C_2 C_5 A(T)^{1-\frac{p}{r\vartheta}} \quad \text{for all } t \in (0, T) \end{aligned}$$

with some $C_5 > 0$, we establish that there exists $C_6 > 0$ such that

$$\|u_2(\cdot, t)\|_{L^\infty(\Omega)} \leq C_6 \quad \text{for all } t \in (0, T). \quad (2.9)$$

Finally, the maximum principle together with the elementary inequality

$$\mu_1 u(1 - u - a_1 v) \leq -\mu_1 \left(u - \frac{1 + \mu_1}{2\mu_1} \right)^2 + \frac{(1 + \mu_1)^2}{4\mu_1} \leq \frac{(1 + \mu_1)^2}{4\mu_1}$$

implies that there exists $C_7 > 0$ such that

$$u_3(\cdot, t) \leq C_7 \quad \text{for all } t \in (0, T). \quad (2.10)$$

Therefore a combination of (2.6), the nonnegativity of u with (2.7), (2.8), (2.9), (2.10) tells us that there exist $C_8, C_9 > 0$ such that

$$A(T) \leq C_8 + C_9 A(T)^{1-\frac{p}{r\vartheta}},$$

which implies from $p < r\vartheta$ that

$$A(T) \leq C_{10} \quad \text{for all } T \in (0, T_{\max})$$

with some $C_{10} > 0$. Thus we obtain the L^∞ -estimate for u . Similarly, we can verify the L^∞ -estimate for v . Then invoking (2.5), we see that there exists $C_{11} > 0$ such that

$$\|w(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C_{11} \quad \text{for all } t \in (0, T_{\max}),$$

which implies (2.3). Moreover, known regularity arguments (see [6, Proposition 2.3]) enable us to find $C_{12} > 0$ and $\theta \in (0, 1)$ satisfying

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} \leq C_{12} \quad \text{for all } t \geq 1,$$

which implies the end of the proof. \square

Proof of Theorem 1.1. Lemma 2.4 directly shows Theorem 1.1. \square

3. Stabilization

In this section we will establish stabilization of solutions to (1.1). Here we assume that there exists a unique global classical solution (u, v, w) of (1.1) satisfying

$$\|u\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|v\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} + \|w\|_{C^{\theta, \frac{\theta}{2}}(\overline{\Omega} \times [t, t+1])} \leq M \quad \text{for all } t \geq 1$$

with some $M > 0$. we first recall a important lemma for the proof of Theorems 1.3 and 1.4 (see [14, Lemma 4.6]).

Lemma 3.1. *Let $n \in C^0(\overline{\Omega} \times [0, \infty))$ satisfy that there exist constants $C^* > 0$ and $\theta^* > 0$ such that*

$$\|n\|_{C^{\theta^*, \frac{\theta^*}{2}}(\overline{\Omega} \times [t, t+1])} \leq C^* \quad \text{for all } t \geq 1.$$

Assume that

$$\int_0^\infty \int_\Omega (n(x, t) - N^*)^2 dx dt < \infty$$

with some constant $N^ > 0$. Then*

$$n(\cdot, t) \rightarrow N^* \quad \text{in } C^0(\overline{\Omega}) \quad \text{as } t \rightarrow \infty.$$

We next provide the following lemma which will be used to confirm that the assumption of Lemma 3.1 is satisfied.

Lemma 3.2. *Let $a, b, c, d, e, f \in \mathbb{R}$. Suppose that*

$$a > 0, \quad d - \frac{b^2}{4a} > 0, \quad f - \frac{c^2}{4a} - \frac{(2ae - bc)^2}{4a(4ad - b^2)} > 0.$$

Then

$$ax^2 + bxy + cxz + dy^2 + eyz + fz^2 \geq 0$$

holds for all $x, y, z \in \mathbb{R}$.

Proof. Straightforward calculations lead to the conclusion of this lemma (for more details, see [27, Lemma 3.2]). \square

Finally, we give the following lemma which enables us to upgrade the L^2 -convergence rate to L^∞ -convergence rate.

Lemma 3.3. *Let $(\bar{u}, \bar{v}, \bar{w}) \in \mathbb{R}^3$ be a solution to (1.1). Assume that there exists a decreasing function $h : [0, \infty) \rightarrow \mathbb{R}$ satisfying*

$$\|u(\cdot, t) - \bar{u}\|_{L^2(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^2(\Omega)} \leq h(t) \quad \text{for all } t > 0.$$

Then there exists $C > 0$ such that

$$\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} + \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} \leq Ch(t-1)^{\frac{1}{n+1}}$$

for all $t > 1$.

Proof. For all $p > 2$ we first obtain from the Hölder inequality that

$$\|f\|_{L^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)}^{1-\frac{2}{p}} \|f\|_{L^2(\Omega)}^{\frac{2}{p}}$$

holds for all $f \in L^\infty(\Omega)$, which means from the boundedness of u, v that

$$\|u(\cdot, t) - \bar{u}\|_{L^p(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^p(\Omega)} \leq C_1(p)h(t)^{\frac{2}{p}} \quad \text{for all } t > 0$$

with some $C_1(p) > 0$. Here (2.5) enables us to see that

$$\|w(\cdot, t) - \bar{w}\|_{W^{2,2n+2}(\Omega)} \leq C_E(2n+2)C_1(2n+2)h(t)^{\frac{1}{n+1}} \quad \text{for all } t > 0.$$

Thus we have that there is $C_2 > 0$ such that

$$\|\nabla w(\cdot, t)\|_{L^{2n+2}(\Omega)} \leq C_2h(t)^{\frac{1}{n+1}} \quad \text{for all } t > 0.$$

Then by using a similar argument as in the proof of [1, Lemma 3.6] we infer that there exists $C_3 > 0$ such that

$$\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)} \leq C_3h(t-1)^{\frac{1}{n+1}} \quad \text{for all } t > 1.$$

Finally, since $(\bar{u}, \bar{v}, \bar{w})$ satisfies

$$\alpha\bar{u} + \beta\bar{v} - \gamma\bar{w} = 0,$$

we can apply the maximum principle to

$$-\Delta(w - \bar{w}) + \gamma(w - \bar{w}) = \alpha(u - \bar{u}) + \beta(v - \bar{v}),$$

and hence obtain the existence of a constant $C_4 > 0$ such that

$$\begin{aligned} \|w(\cdot, t) - \bar{w}\|_{L^\infty(\Omega)} &\leq C_4(\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{L^\infty(\Omega)}) \\ &\leq C_3C_4h(t-1)^{\frac{1}{n+1}} \quad \text{for all } t > 1, \end{aligned}$$

which concludes the proof of this lemma. \square

3.1. Convergence. Case 1: $a_1, a_2 \in (0, 1)$

In this subsection we establish stabilization in the case that $a_1, a_2 \in (0, 1)$. We first confirm that the assumption of Lemma 3.1 are satisfied.

Lemma 3.4. *Assume that (1.8)–(1.10) are satisfied. Then there exist a nonnegative function $E_1 : (0, \infty) \rightarrow \mathbb{R}$ and a constant $\varepsilon > 0$ such that*

$$\frac{d}{dt}E_1(t) \leq -\varepsilon \int_{\Omega} [(u(\cdot, t) - u^*)^2 + (v(\cdot, t) - v^*)^2 + (w(\cdot, t) - w^*)^2] \quad (3.1)$$

holds for all $t > 0$. Moreover, there exists $C > 0$ satisfying

$$\int_0^\infty \int_{\Omega} (u - u^*)^2 + \int_0^\infty \int_{\Omega} (v - v^*)^2 + \int_0^\infty \int_{\Omega} (w - w^*)^2 \leq C.$$

Proof. Let $\delta_1 > 0$ be a constant defined in (1.8)–(1.10). First we shall show that the function $E_1 : (0, \infty) \rightarrow \mathbb{R}$ defined as

$$E_1 := \int_{\Omega} \left(u - u^* - u^* \log \frac{u}{u^*} \right) + \frac{a_1 \mu_1 \delta_1}{a_2 \mu_2} \int_{\Omega} \left(v - v^* - v^* \log \frac{v}{v^*} \right) \quad (3.2)$$

satisfies that (3.1) holds for all $t > 0$ with some $\varepsilon > 0$. From straightforward calculations we infer that

$$\begin{aligned} \frac{d}{dt}E_1(t) = & -\mu_1 \int_{\Omega} (u(\cdot, t) - u^*)^2 - (1 + \delta_1) a_1 \mu_1 \int_{\Omega} (u(\cdot, t) - u^*)(v(\cdot, t) - v^*) \\ & - \frac{a_1 \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - v^*)^2 - d_1 u^* \int_{\Omega} \frac{|\nabla u(\cdot, t)|^2}{u^2} \\ & + u^* \chi_1 \int_{\Omega} \frac{\nabla u(\cdot, t) \cdot \nabla w(\cdot, t)}{u} - \frac{d_2 a_1 \mu_1 v^* \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{|\nabla v(\cdot, t)|^2}{v^2} \\ & + \frac{a_1 \mu_1 v^* \chi_2 \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{\nabla v(\cdot, t) \cdot \nabla w(\cdot, t)}{v} \quad \text{for all } t > 0. \end{aligned} \quad (3.3)$$

Here in light of (1.8)–(1.10) we can take $\delta_2 > 0$ satisfying

$$\frac{u^* \chi_1^2}{4d_1} < \delta_2 < \frac{d_3 a_1 \mu_1 \gamma (4\delta_1 - (1 + \delta_1)^2 a_1 a_2)}{(1 + \delta_1)(a_1 \alpha^2 \delta_1 + a_2 \beta^2 - (1 + \delta_1) a_1 a_2 \alpha \beta)}$$

and

$$\frac{a_1 \mu_1 v^* \chi_2^2}{4d_2 a_2 \mu_2} < \delta_2 < \frac{d_3 a_1 \mu_1 \gamma (4\delta_1 - (1 + \delta_1)^2 a_1 a_2)}{(1 + \delta_1)(a_1 \alpha^2 \delta_1 + a_2 \beta^2 - (1 + \delta_1) a_1 a_2 \alpha \beta)}.$$

Invoking the Young inequality, we obtain that

$$u^* \chi_1 \int_{\Omega} \frac{\nabla u \cdot \nabla w}{u} \leq \frac{u^{*2} \chi_1^2}{4\delta_2} \int_{\Omega} \frac{|\nabla u|^2}{u^2} + \delta_2 \int_{\Omega} |\nabla w|^2 \quad (3.4)$$

and

$$\frac{a_1\mu_1v^*\chi_2\delta_1}{a_2\mu_2} \int_{\Omega} \frac{\nabla v \cdot \nabla w}{v} \leq \frac{a_1^2\mu_1^2v^{*2}\chi_2^2\delta_1}{4\delta_2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \delta_1\delta_2 \int_{\Omega} |\nabla w|^2. \quad (3.5)$$

Therefore since the definition of δ_2 yields

$$d_1 - \frac{u^*\chi_1^2}{4\delta_2} > 0 \quad \text{and} \quad d_2 - \frac{a_1\mu_1v^*\chi_2^2}{4a_2\mu_2\delta_2} > 0,$$

a combination of (3.3) with (3.4) and (3.5) implies

$$\begin{aligned} \frac{d}{dt}E_1(t) &\leq -\mu_1 \int_{\Omega} (u(\cdot, t) - u^*)^2 - (1 + \delta_1)a_1\mu_1 \int_{\Omega} (u(\cdot, t) - u^*)(v(\cdot, t) - v^*) \\ &\quad - \frac{a_1\mu_1\delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - v^*)^2 + (1 + \delta_1)\delta_2 \int_{\Omega} |\nabla w(\cdot, t)|^2 \quad \text{for all } t > 0. \end{aligned}$$

Noting from the third equation in (1.1) that

$$\int_{\Omega} |\nabla w|^2 = \frac{\alpha}{d_3} \int_{\Omega} (u - u^*)(w - w^*) + \frac{\beta}{d_3} \int_{\Omega} (v - v^*)(w - w^*) - \frac{\gamma}{d_3} \int_{\Omega} (w - w^*)^2,$$

we establish that

$$\frac{d}{dt}E_1(t) \leq F_1(t) \quad \text{for all } t > 0,$$

where

$$\begin{aligned} F_1(t) &:= -\mu_1 \int_{\Omega} (u(\cdot, t) - u^*)^2 - (1 + \delta_1)a_1\mu_1 \int_{\Omega} (u(\cdot, t) - u^*)(v(\cdot, t) - v^*) \\ &\quad - \frac{a_1\mu_1\delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - v^*)^2 + \frac{\alpha(1 + \delta_1)\delta_2}{d_3} \int_{\Omega} (u(\cdot, t) - u^*)(w(\cdot, t) - w^*) \\ &\quad + \frac{\beta(1 + \delta_1)\delta_2}{d_3} \int_{\Omega} (v(\cdot, t) - v^*)(w(\cdot, t) - w^*) - \frac{\gamma(1 + \delta_1)\delta_2}{d_3} \int_{\Omega} (w(\cdot, t) - w^*)^2 \end{aligned}$$

for all $t > 0$. In order to see (3.1) we will show that

$$F_1(t) \leq -\varepsilon \int_{\Omega} [(u(\cdot, t) - u^*)^2 + (v(\cdot, t) - v^*)^2 + (w(\cdot, t) - w^*)^2]$$

with some $\varepsilon > 0$ by using Lemma 3.2. To confirm that the assumption of Lemma 3.2 is satisfied we put

$$\begin{aligned} g_1(\varepsilon) &:= \mu_1 - \varepsilon, \quad g_2(\varepsilon) := \frac{a_1\mu_1\delta_1}{a_2} - \varepsilon - \frac{(1 + \delta_1)^2 a_1^2 \mu_1^2}{4(\mu_1 - \varepsilon)}, \\ g_3(\varepsilon) &:= \frac{\gamma(1 + \delta_1)\delta_2}{d_3} - \varepsilon - \frac{\alpha^2(1 + \delta_1)^2 \delta_2^2}{4d_3^2(\mu_1 - \varepsilon)} \\ &\quad - \frac{(2(\mu_1 - \varepsilon)\beta(1 + \delta_1)\delta_2 - (1 + \delta_1)^2 a_1\mu_1\alpha\delta_1)^2}{4d_3^2(\mu_1 - \varepsilon) \left(4(\mu_1 - \varepsilon) \left(\frac{a_1\mu_1\delta_1}{a_2} - \varepsilon \right) - (1 + \delta_1)^2 a_1\mu_1 \right)} \end{aligned}$$

for $\varepsilon > 0$, and shall see that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ for $i = 1, 2, 3$. Here $g_1(0) = \mu_1 > 0$ obviously holds, and the condition (1.8) implies that

$$g_2(0) = \frac{a_1\mu_1(4\delta_1 - (1 + \delta_1)^2 a_1 a_2)}{4a_2} > 0.$$

Moreover, aided by the definition of δ_2 , we can obtain that

$$\begin{aligned} g_3(0) &= \frac{\gamma(1 + \delta_1)\delta_2}{d_3} - \frac{\alpha^2(1 + \delta_1)^2\delta_2^2}{4d_3^2\mu_1} - \frac{(2\mu_1\beta(1 + \delta_1)\delta_2 - (1 + \delta_1)^2 a_1\mu_1\alpha\delta_1)^2}{4d_3^2\mu_1 \left(4\mu_1 \frac{a_1\mu_1\delta_1}{a_2} - (1 + \delta_1)^2 a_1\mu_1\right)} \\ &= (1 + \delta_1)\delta_2 \left(\frac{\gamma}{d_3} - \frac{(1 + \delta_1)(\alpha^2 a_1\delta_1 + a_2\beta^2 - (1 + \delta_1)a_1 a_2\alpha\beta)}{d_3^2 a_1\mu_1(4\delta_1 - (1 + \delta_1)^2 a_1 a_2)} \delta_2 \right) > 0. \end{aligned}$$

Therefore a combination of the above inequalities and the continuity argument implies that there exists $\varepsilon_1 > 0$ such that $g_i(\varepsilon_1) > 0$ hold for $i = 1, 2, 3$. Thus Lemma 3.2 derives that

$$F_1(t) \leq -\varepsilon_1 \int_{\Omega} [(u(\cdot, t) - u^*)^2 + (v(\cdot, t) - v^*)^2 + (w(\cdot, t) - w^*)^2] \quad \text{for all } t > 0,$$

which yields that (3.1) holds for all $t > 0$. Then since from the Taylor formula E_1 is a nonnegative function for $t > 0$ (more details, see [1, Lemma 3.2]), integrating (3.1) over $(0, \infty)$ concludes the proof of this lemma. \square

Lemma 3.5. *Assume that (1.8)–(1.10) are satisfied. Then*

$$\|u(\cdot, t) - u^*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v^*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w^*\|_{L^\infty(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. A combination of Lemmas 3.1 and 3.4 implies this lemma. \square

Next we desire to establish convergence rates for the solution of (1.1). We note that in view of Lemma 3.3 it is sufficient to confirm the L^2 -convergence rates for the solution.

Lemma 3.6. *Assume that (1.8)–(1.10) are satisfied. Then there exist $C > 0$ and $\ell > 0$ such that*

$$\|u(\cdot, t) - u^*\|_{L^2(\Omega)} + \|v(\cdot, t) - v^*\|_{L^2(\Omega)} \leq Ce^{-\ell t} \quad \text{for all } t > 0.$$

Proof. Aided by Lemma 3.5 and the L'Hôpital theorem, a similar argument as in the proof of [1, Lemma 3.7] (or [27, Proof of Theorem 1.2]) derives that there exist $C_1, C_2 > 0$ and $t_0 > 0$ such that for all $t > t_0$,

$$C_1 \left(\int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 \right) \leq E_1 \leq C_2 \left(\int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 \right), \quad (3.6)$$

where E_1 is the function defined as (3.2). Therefore we obtain from (3.1) that

$$\frac{d}{dt} E_1(t) \leq -C_3 E_1(t) \quad \text{for all } t > t_0$$

with some $C_3 > 0$, which implies that there exists $C_4 > 0$ such that

$$E_1(t) \leq C_4 e^{-C_3 t} \quad \text{for all } t > 0. \quad (3.7)$$

Thus a combination of (3.6) and (3.7) yields that

$$\int_{\Omega} (u - u^*)^2 + \int_{\Omega} (v - v^*)^2 \leq \frac{C_4}{C_1} e^{-C_3 t},$$

which concludes the proof of this lemma. \square

3.2. Convergence. Case 2: $a_1 \geq 1 > a_2$

In this subsection we will obtain stabilization in the case that $a_1 \geq 1 > a_2$. In this case we also have to confirm that the assumption of Lemma 3.1 is satisfied.

Lemma 3.7. *Assume that (1.11)–(1.12) are satisfied. Then there exist a nonnegative function $E_2 : (0, \infty) \rightarrow \mathbb{R}$ and constants $\varepsilon_1 \geq 0$ and $\varepsilon_2 > 0$ such that*

$$\frac{d}{dt} E_2(t) \leq -\varepsilon_1 \int_{\Omega} u(\cdot, t) - \varepsilon_2 \int_{\Omega} \left[u(\cdot, t)^2 + (v(\cdot, t) - 1)^2 + \left(w(\cdot, t) - \frac{\beta}{\gamma} \right)^2 \right] \quad (3.8)$$

holds for all $t > 0$. Moreover, there exists $C > 0$ satisfying

$$\int_0^\infty \int_{\Omega} u^2 + \int_0^\infty \int_{\Omega} (v - 1)^2 + \int_0^\infty \int_{\Omega} \left(w - \frac{\beta}{\gamma} \right)^2 \leq C.$$

Proof. Let $\delta_1 > 0$ and $a'_1 \in [1, a_1]$ be a constant defined in (1.11)–(1.12). We first show that the function $E_2 : (0, \infty) \rightarrow \mathbb{R}$ defined as

$$E_2 := \int_{\Omega} u + \frac{a'_1 \mu_1 \delta_1}{a_2 \mu_2} \int_{\Omega} (v - 1 - \log v) \quad (3.9)$$

fulfils (3.8) for all $t > 0$ with $\varepsilon_1 := a'_1 - 1$ and some $\varepsilon_2 > 0$. Noting from the relation $a'_1 \leq a_1$ that

$$\begin{aligned} u(1 - u - a_1 v) &\leq u(1 - u - a'_1 v) \\ &= -\mu_1 u^2 - a'_1 \mu_1 u(v - 1) - (a'_1 - 1) \mu_1 u, \end{aligned}$$

from straightforward calculations we derive that

$$\begin{aligned} \frac{d}{dt} E_2(t) &= -\mu_1 \int_{\Omega} u(\cdot, t)^2 - (1 + \delta_1) a'_1 \mu_1 \int_{\Omega} u(\cdot, t)(v(\cdot, t) - 1) \\ &\quad - \frac{a'_1 \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - 1)^2 - (a'_1 - 1) \int_{\Omega} u(\cdot, t) \\ &\quad - \frac{d_2 a'_1 \mu_1 \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{|\nabla v(\cdot, t)|^2}{v^2} + \frac{a'_1 \mu_1 \chi_2 \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{\nabla v(\cdot, t) \cdot \nabla w(\cdot, t)}{v} \end{aligned} \quad (3.10)$$

for all $t > 0$. Here thanks to (1.11)–(1.12), we can take $\delta_2 > 0$ such that

$$\frac{a'_1 \mu_1 \chi_2^2 \delta_1}{4d_2 a_2 \mu_2} < \delta_2 < \frac{d_3 a'_1 \mu_1 \gamma (4\delta_1 - (1 + \delta_1)^2 a'_1 a_2)}{a'_1 \alpha^2 \delta_1 + a_2 \beta^2 - (1 + \delta_1) a'_1 a_2 \alpha \beta}.$$

Invoking the Young inequality, we obtain that

$$\frac{a'_1 \mu_1 \chi_2 \delta_1}{a_2 \mu_2} \int_{\Omega} \frac{\nabla v \cdot \nabla w}{v} \leq \frac{a_1'^2 \mu_1^2 \chi_2^2 \delta_1^2}{4\delta_2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} + \delta_2 \int_{\Omega} |\nabla w|^2.$$

Therefore since the definition of δ_2 yields

$$d_2 - \frac{a'_1 \mu_1 \chi_2^2 \delta_1}{4a_2 \mu_2 \delta_2} > 0,$$

the equation (3.10) implies that

$$\begin{aligned} \frac{d}{dt} E_2(t) &\leq -\varepsilon_1 \int_{\Omega} u - \mu_1 \int_{\Omega} u(\cdot, t)^2 - (1 + \delta_1) a'_1 \mu_1 \int_{\Omega} u(\cdot, t)(v(\cdot, t) - 1) \\ &\quad - \frac{a'_1 \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - 1)^2 + \delta_2 \int_{\Omega} |\nabla w(\cdot, t)|^2 \end{aligned}$$

for all $t > 0$, where $\varepsilon_1 = a'_1 - 1$. Noting from the third equation in (1.1) that

$$\int_{\Omega} |\nabla w|^2 = \frac{\alpha}{d_3} \int_{\Omega} u \left(w - \frac{\beta}{\gamma} \right) + \frac{\beta}{d_3} \int_{\Omega} (v - 1) \left(w - \frac{\beta}{\gamma} \right) - \frac{\gamma}{d_3} \int_{\Omega} \left(w - \frac{\beta}{\gamma} \right)^2,$$

we establish that for all $t > 0$,

$$\frac{d}{dt} E_2(t) \leq -\varepsilon_1 \int_{\Omega} u + F_2(t) \quad \text{for all } t > 0,$$

where

$$\begin{aligned} F_2(t) &:= -\mu_1 \int_{\Omega} u(\cdot, t)^2 - (1 + \delta_1) a_1 \mu_1 \int_{\Omega} u(\cdot, t)(v(\cdot, t) - 1) \\ &\quad - \frac{a_1 \mu_1 \delta_1}{\mu_2} \int_{\Omega} (v(\cdot, t) - 1)^2 + \frac{\alpha(1 + \delta_1) \delta_2}{d_3} \int_{\Omega} u(\cdot, t) \left(w(\cdot, t) - \frac{\beta}{\gamma} \right) \\ &\quad + \frac{\beta(1 + \delta_1) \delta_2}{d_3} \int_{\Omega} (v(\cdot, t) - 1) \left(w(\cdot, t) - \frac{\beta}{\gamma} \right) - \frac{\gamma(1 + \delta_1) \delta_2}{d_3} \int_{\Omega} \left(w(\cdot, t) - \frac{\beta}{\gamma} \right)^2. \end{aligned}$$

Then by using the same argument as in the proof of Lemma 3.4 we can see that

$$F_2(t) \leq -\varepsilon_2 \int_{\Omega} \left[u(\cdot, t)^2 + (v(\cdot, t) - 1)^2 + \left(w(\cdot, t) - \frac{\beta}{\gamma} \right)^2 \right] \quad \text{for all } t > 0$$

with some $\varepsilon_2 > 0$, which means that (3.8) holds with $\varepsilon_1 = a'_1 - 1 \geq 0$ and $\varepsilon_2 > 0$. \square

Then we will establish the convergence result for the solution to (1.1) in the case that $a_1 \geq 1 > a_2$.

Lemma 3.8. *Assume that (1.11)–(1.12) are satisfied. Then we have*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - 1\|_{L^\infty(\Omega)} + \left\| w(\cdot, t) - \frac{\beta}{\gamma} \right\|_{L^\infty(\Omega)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Proof. A combination of Lemmas 3.1 and 3.7 implies this lemma. \square

Finally, we shall show two lemmas which give asymptotic behavior in the case that $a_1 > 1 > a_2$.

Lemma 3.9. *Let $a_1 > 1$ and $a_2 \in (0, 1)$. Assume that (1.11)–(1.12) are satisfied with $\delta_1 > 0$ and $a'_1 > 0$. Then there exist $C > 0$ and $\ell > 0$ satisfying*

$$\|u(t)\|_{L^2(\Omega)} + \|v(t) - 1\|_{L^2(\Omega)} \leq Ce^{-\ell t} \quad \text{for all } t > 0.$$

Proof. In the case that $a_1 \geq 1$ and $a_2 \in (0, 1)$ a similar argument as in the proof of [27, Lemmas 4.3] enables us to see that there exist $C_1, C_2 > 0$ and $t_0 > 0$ such that

$$C_1 h_1(t) \leq E_2(t) \leq C_2 h_1(t) \quad \text{for all } t > t_0,$$

where E_2 is the function defined as (3.9) and

$$h_1(t) := \int_{\Omega} u(\cdot, t)^2 + \int_{\Omega} (v(\cdot, t) - 1)^2 + (a'_1 - 1) \int_{\Omega} u(\cdot, t).$$

Thus a combination of the above inequality and (3.8) means that

$$\frac{d}{dt} E_2(t) \leq -C_3 E_2(t)$$

holds for all $t > t_0$, which together with the same argument as in the proof of Lemma 3.6 leads to the conclusion of this lemma in the case that $a_1 > 1$ and $a_2 \in (0, 1)$. \square

Lemma 3.10. *Let $a_1 = 1$ and $a_2 \in (0, 1)$. Assume that (1.11)–(1.12) are satisfied. Then there exist $C > 0$ and $\ell > 0$ satisfying*

$$\|u(t)\|_{L^2(\Omega)} + \|v(t) - 1\|_{L^2(\Omega)} \leq \frac{C}{\sqrt{t+2}} \quad \text{for all } t > 0.$$

Proof. First we can verify from the same argument as in the proof of [27, Lemma 3.7] that there exist $C_4, C_5 > 0$ and $t_1 > 0$ such that

$$C_4 \int_{\Omega} (v(\cdot, t) - 1)^2 \leq \int_{\Omega} (v(\cdot, t) - 1 - \log v(\cdot, t)) \leq C_5 \int_{\Omega} (v(\cdot, t) - 1)^2 \quad (3.11)$$

for all $t > t_1$. Hence it follows from the Cauchy–Schwarz inequality and the boundedness of v that

$$\begin{aligned} E_2(t) &\leq \int_{\Omega} u(\cdot, t) + \frac{a'_1 \mu_1 \delta_1}{a_2 \mu_2} \int_{\Omega} (v(\cdot, t) - 1)^2 \\ &\leq C_6 \left(\int_{\Omega} u(\cdot, t)^2 \right)^{\frac{1}{2}} + C_6 \left(\int_{\Omega} (v(\cdot, t) - 1)^2 \right)^{\frac{1}{2}} \\ &\leq \sqrt{2} C_6 \left(\int_{\Omega} u(\cdot, t)^2 + \int_{\Omega} (v(\cdot, t) - 1)^2 \right)^{\frac{1}{2}} \quad \text{for all } t > t_1, \end{aligned}$$

which implies from (3.8) that

$$E_2(t) \leq -C_7 E_2(t)^2 \quad \text{for all } t > t_1.$$

Therefore we can find $C_8 > 0$ such that

$$E_2(t) \leq \frac{C_8}{t+2} \quad \text{for all } t > t_1.$$

Therefore thanks to the boundedness of u and (3.11), we obtain that

$$\int_{\Omega} u(\cdot, t)^2 + \int_{\Omega} (v(\cdot, t) - 1)^2 \leq C_9 E_2(t) \leq \frac{C_8 C_9}{t+2} \quad \text{for all } t > t_1,$$

which proves this lemma. \square

Proof of Theorems 1.3 and 1.4. A combination of Lemmas 3.6, 3.9, 3.10 and 3.3 immediately leads to the conclusions of these theorems. \square

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